Overall Exam Problems for Algebra, Number Theory and Combinatorics

1. Show that any finite abelian group is the Galois group of some field extension of $\mathbb{Q}$. Give an example of a finite extension of $\mathbb{Q}$ with non-abelian Galois group.

2. Let $p > 2$ be a prime number and F_p the finite field with p elements.

i) Determine the order of the group $\mathrm{SL}_2(F_p)$.

ii) How many Sylow p-subgroups are there in $\mathrm{SL}_2(F_p)$?

iii) Show that there exists an element $\mu \in F_p$ which is not a square. For $a, b \in F_p$ with $a^2 - b^2\mu = 1$, consider the following element in $\mathrm{SL}_2(F_p)$:

$$M_{a,b} := \begin{pmatrix} a & b\mu \\ b & a \end{pmatrix}.$$

Show that $M_{a,b}$ is conjugate to $M_{a',b'}$ in $\mathrm{SL}_2(F_p)$ if and only if $a = a'$ and $b = \pm b'$.

iv) How many conjugacy classes of the form $M_{a,b}$ are there in $\mathrm{SL}_2(F_p)$?